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LETTER TO THE EDITOR

Hecke algebra solutions to the reflection equation

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Abstract. We construct solutions to Sklyanin's reflection equation in the case in which the bulk Yang–Baxter solution is of Hecke algebra type. Each solution constitutes an extension of the Hecke algebra together with a spectral parameter dependent boundary operator, $K(\theta)$. We solve for the defining relations of the extension, and for the spectral parameter dependence of the boundary operator. The problem of finding concrete matrix solutions to the reflection equation is thus reduced to that of constructing finite dimensional representations of the Hecke algebra extension. This problem is solved in the generic case by establishing isomorphism with an affine Hecke algebra quotient whose representation theory is known.

In 2D lattice statistical mechanics it is found (see Baxter [1]) that the Yang–Baxter equations:

$$R_i(\theta_1)R_{i+1}(\theta_1+\theta_2)R_i(\theta_2) = R_{i+1}(\theta_2)R_i(\theta_1+\theta_2)R_{i+1}(\theta_1) \tag{1}$$

$$[R_i(\theta_1), R_j(\theta_2)] = 0 \qquad i \neq j \pm 1 \tag{2}$$

serve as integrability conditions. However, Baxter's method of constructing commuting transfer matrices from a given solution of (1) and (2) requires periodic or quasi-periodic boundary conditions. In [2] Sklyanin proposed an ingenious extension of Baxter's formalism which reconciles integrability with non-periodic boundaries—for a given solution of equations (1) and (2) one looks for a boundary operator $K(\theta)$ satisfying the equations:

$$R_1(\theta_1-\theta_2)K(\theta_1)R_1(\theta_1+\theta_2)K(\theta_2) = K(\theta_2)R_1(\theta_1+\theta_2)K(\theta_1)R_1(\theta_1-\theta_2) \tag{3}$$

$$[R_i(\theta'), K(\theta)] = 0 \qquad i \geqslant 2. \tag{4}$$

Solutions to (1)–(4) may lead to an integrable model whose boundary conditions are encoded in $K(\theta)$. For Sklyanin's scheme to work, $R_i(\theta)$ and $K(\theta)$ have to satisfy some additional conditions. In particular, the 'unitarity condition on $K(\theta)$' which plays a role in the present work requires:

$$K(\theta)K(-\theta) = \phi(\theta)1 \qquad (\phi \text{ scalar}). \tag{5}$$

While many solutions to (1) and (2) are known, the study of (3) and (4) was less extensive (but see [2–6] which also discuss physical properties of some of the resulting models). Recently, Zamolodchikov and Ghoshal [7] discussed the application of equations (1)–(5) to 2D integrable S-matrix theories. In the scattering picture, $R_i(\theta)$ represents a 2-particle scattering in the bulk, while $K(\theta)$ represents a reflection from a wall at the boundary [3], and so (3) is called the *reflection equation* (RE). They point out that the collection of all integrable boundary scattering theories which have the same bulk scattering properties is an interesting object to study (see [7] for a full discussion and motivation). Here we study solutions to (3)–(5) for a given solution of (1) and (2) in a specific algebraic framework. We solve for $K(\theta)$ in cases for which $R_i(\theta)$ is given in terms of Hecke algebra generators:

For q a scalar parameter (we will also use $q = \exp(i\gamma)$), we denote by $H_n = H_n(q)$ the A_n Hecke algebra over field $\mathbb{C}(q)$ [10, 12] which is generated by $\{1, U_1, U_2, \ldots, U_{n-1}\}$ satisfying:

$$\begin{aligned} U_iU_i &= (q+q^{-1})U_i \\ U_iU_{i+1}U_i - U_i &= U_{i+1}U_iU_{i+1} - U_{i+1} \\ [U_i, U_j] &= 0 \qquad |i-j| > 1. \end{aligned}$$

The 'Hecke bulk solution' of the Yang–Baxter equation (YBE) (1) is then

$$R_i(\theta) = \sin(\gamma+\theta)1 - \sin(\theta)U_i. \tag{6}$$

Any representation of H_n gives a matrix solution to (1) and (2) via (6). Substituting (6) into (3) we get

$$\begin{aligned} \sin(\theta_1+\theta_2)\sin(\theta_1-\theta_2)[U_1K(\theta_1)U_1, K(\theta_2)] &= \sin(\gamma+\theta_1-\theta_2)\sin(\theta_1+\theta_2)(K(\theta_1)U_1K(\theta_2) \\ &\quad - K(\theta_2)U_1K(\theta_1)) + \sin(\gamma+\theta_1+\theta_2)\sin(\theta_1-\theta_2)(U_1K(\theta_1)K(\theta_2) \\ &\quad - K(\theta_2)K(\theta_1)U_1) - \sin(\gamma+\theta_1+\theta_2)\sin(\gamma+\theta_1-\theta_2)[K(\theta_1), K(\theta_2)]. \end{aligned} \tag{7}$$

Let χ be an associative algebra over $\mathbb{C}$ with basis $\{\chi_1, \chi_2, \ldots, \chi_d\}$. An algebraic solution to (1)–(5) is one in which $K(\theta)$ takes values in such a χ (cf equation (6)):

$$K(\theta) = \sum_{\alpha=1}^{d} f_\alpha(\theta)\chi_\alpha \qquad (f_\alpha \text{ scalars}). \tag{8}$$

To satisfy (4) we require $[U_i, \chi] = 0$ for $i > 1$. Inserting (8) into (7) and (5) we look for solutions in the form of a set of consistent θ independent relations between U_1 and χ together with constraints on the functions $f_\alpha(\theta)$. The relations will define a new algebra A which is an extension of H_n by χ. Thus, a Hecke algebra solution to (1–5) consists of an algebra A and a function $K(\theta) \in A$.

Such $(A, K(\theta))$ pairs are given for $d = 2$ in the following section and for arbitrary d in theorem 1. Note that stronger algebraic relations allow relatively weaker constraints on the functions (e.g. (7) is solved by $[U_1, \chi] = 0$ and χ commutative—we will call this the *trivial* solution). Here we look for a *minimal* set of relations which are sufficient for a solution, and determine $f_\alpha(\theta)$ for this set. Matrix solutions to (3)–(5) will thus arise from matrix representations of the H_n extension. Next we facilitate the complete characterization of finite dimensional representations in the generic case by establishing an isomorphism with a known affine Hecke quotient. We conclude with a discussion of outstanding problems.

For $d = 1$ we get $\chi = \mathbb{C}$ and $K(\theta) = f_1(\theta)1$ solves RE for any $f_1(\theta)$. Unitarity gives $f_1(\theta)f_1(-\theta) = \phi(\theta)$. The case $d = 2$ better illustrates our programme, so we start with this.

We will now consider the case $d = 2$.

All possibilities here are encapsulated by $\chi(b, c) \equiv \langle 1, X : X^2 = bX + c1\rangle$ where b and c are scalar parameters. We get:

$$K(\theta) = f_1(\theta)1 + f_2(\theta)X. \tag{9}$$

An immediate consequence of (9) is that $[K(\theta_1), K(\theta_2)] = 0$ for all θ_1, θ_2. We assume that $f_2(\theta)$ is not identically zero, otherwise we are dealing with a $d = 1$ case. Then (7) gives:

$$\begin{aligned}[U_1, X](\sin\gamma(f_2(\theta_1)f_1(\theta_2)\sin(2\theta_2) - f_1(\theta_1)f_2(\theta_2)\sin(2\theta_1)) - b\sin(\theta_1 - \theta_2)\\ \times\sin(\theta_1 + \theta_2 + \gamma)f_2(\theta_1)f_2(\theta_2)) + (U_1XU_1X - XU_1XU_1)f_2(\theta_1)f_2(\theta_2)\\ \times\sin(\theta_1 - \theta_2)\sin(\theta_1 + \theta_2) = 0.\end{aligned} \tag{10}$$

We have two possibilities—the trivial solution, or $[U_1, X] \neq 0$. In the latter case, since $f_2(\theta) \neq 0$ we must assume a linear relation between $(U_1XU_1X - XU_1XU_1)$ and $[U_1, X]$. We parameterize this relation in the following way:

$$(U_1XU_1X - XU_1XU_1) = (b\cos\gamma + k\sin\gamma)[U_1, X] \tag{11}$$

where k is θ independent. The structure of the resulting Hecke extension is known, so it only remains to solve for $f_\alpha(\theta)$. Inserting (11) into (10) and dividing by $f_1(\theta_1)f_1(\theta_2)$ we get

$$\frac{f_2(\theta)}{f_1(\theta)} = -\frac{2\sin(2\theta)}{k\cos(2\theta) + b\sin(2\theta) + f} \qquad (f \text{ any constant}). \tag{12}$$

Taking $\phi = 1$ the unitarity equation (5) gives:

$$f_1(\theta)f_1(-\theta) + cf_2(\theta)f_2(-\theta) = 1 \tag{13}$$

$$f_1(\theta)f_2(-\theta) + f_1(-\theta)f_2(\theta) + bf_2(\theta)f_2(-\theta) = 0. \tag{14}$$

Dividing both equations by $f_1(\theta)f_1(-\theta)$ we find that (14) is automatically satisfied by (12), while (13) can be solved for $f_1(\theta)$, e.g.

$$f_1(\theta) = \frac{k\cos(2\theta) + b\sin(2\theta) + f}{k\cos(2\theta) + \sqrt{b^2 + 4c}\sin(2\theta) + f}. \tag{15}$$

Note that (15) is non-unique [7]. For example, if $f_1(\theta)$ is any solution to (13), then $f_1(\theta)h(\theta)$ is also a solution provided that $h(\theta)h(-\theta) = 1$.

Now we look at higher d solutions.

We now give solutions for commutative semi-simple χ, of arbitrary dimension d. Let χ^d be the d-dimensional commutative semi-simple $\mathbb{C}$-algebra, i.e. $\chi^d = \mathbb{C} \oplus \mathbb{C} \oplus \cdots \oplus \mathbb{C}$ (d summands). Let v_α be the unique projection on the αth summand of χ^d, so that $v_\alpha v_\beta = \delta_{\alpha\beta}v_\beta$ and $\sum_{\alpha=1}^{d} v_\alpha = 1$. Then a convenient basis for χ^d is $\{v_1, v_2, \ldots, v_d\}$. Let $\{w_1(\theta), w_2(\theta), \ldots, w_d(\theta)\}$ be the eigenvalues of $K(\theta) \in \chi^d$, then

$$K(\theta) = \sum_{\alpha=1}^{d} w_\alpha(\theta)v_\alpha. \tag{16}$$

We will say that $K(\theta)$ is a degenerate if it may be contained in a $\chi^{d'}$ subalgebra of $\chi^d (d' < d)$. Note that this happens if and only if $K(\theta)$ has degenerate eigenvalues. Note that the $K(\theta)$ given in (16) solves the unitarity equation (5) with $\phi = 1$ iff

$$w_\alpha(\theta) w_\alpha(-\theta) = 1 \qquad \forall\alpha. \tag{17}$$

Definition 1 (Hecke algebra extension). Let q be an indeterminate, let a_- be a $d(d-1)/2$-tuple of indeterminates $a_{\alpha\beta} (1 \leqslant \alpha < \beta \leqslant d)$, and set $a_{\alpha\beta} = 2\cos\gamma - a_{\beta\alpha}$ for $\alpha > \beta$. Then $BH_n^d = BH_n(q, a_-)$ is the $\mathbb{C}(q, a_-)$ algebra generated by $\langle H_n(q), \chi^d \rangle / \sim$ where $\sim$ is:

$$[\chi^d, U_i] = 0 \qquad \forall i > 1 \tag{18}$$

$$[U_1 v_\alpha U_1, v_\beta] = a_{\alpha\beta}(v_\alpha U_1 v_\beta - v_\beta U_1 v_\alpha) \equiv a_{\alpha\beta} L_{\alpha\beta} \qquad (\alpha \neq \beta). \tag{19}$$

A complete specialization of BH_n^d is one in which the subring $\mathbb{C}[q, a_-] \to \mathbb{C}$. In certain such specializations (we discuss existence shortly) we can construct a $K(\theta) \in \chi^d$ which solves the reflection and unitarity equations. This construction is our main result (whose proof is by explicit calculation):

Theorem 1. Let A be some complete specialization of BH_n^d and $w_1(\theta)$ a solution of (17) for $\alpha = 1$. Then $(A, K(\theta))$ is a non-degenerate solution of (1)–(5) provided that either:

1. If $\sin\gamma = 0$, then $a_{\alpha\beta} = \cos\gamma$ for all α, β, and $w_\alpha(\theta)$ obeys (17) $\forall\alpha$.
2. If $\sin\gamma \neq 0$, then $(A, K(\theta))$ belongs to one of the following classes. Introduce $c_{\alpha\beta} = a_{\alpha\beta} - \cos\gamma$, $c_\alpha = c_{1\alpha}$, and let $\delta, \epsilon \in \{-1, 1\}$. Then each decreasing sequence $S = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ with $\alpha_i \in \{2, 3, \ldots, d\}$ and choice (δ, ϵ) determines a class of solutions with d continuous parameters. To obtain a specific solution in a given class we may choose any values for the k parameters c_λ with $\lambda \in S$, provided that they are distinct, and distinct from $c_0 = \delta \cdot (i \sin\gamma)$, any set of distinct values for $d - 1 - k$ parameters $d_\alpha (\alpha \in \{2, 3, \ldots, d\}, \alpha \notin S)$ and any value for a parameter d_{α_k}. Then A is determined as follows, where $\alpha, \beta, \lambda, \mu \neq 1$, $\alpha, \beta \notin S$ and one or both of $\lambda, \mu \in S$:

$$c_{\alpha\beta} = c_0 \frac{(d_\alpha + d_\beta)}{d_\alpha - d_\beta} \qquad c_{\lambda\mu} = \frac{c_\lambda c_\mu + \sin^2\gamma}{c_\lambda - c_\mu} \qquad c_\alpha = c_0 \tag{20}$$

and $K(\theta)$ is determined by

$$y_\alpha(\theta) = \frac{w_\alpha(\theta)}{w_1(\theta)} = \frac{c_\alpha \cos(2\theta) + \sin(\gamma)\sin(2\theta) + d_\alpha}{c_\alpha \cos(2\theta) - \sin(\gamma)\sin(2\theta) + d_\alpha} \qquad \forall\alpha \in \{2, \ldots, d\} \tag{21}$$

where for $\lambda \in S$

$$d_\lambda = \frac{d_{\alpha_k}(\sin^2\gamma + c_{\alpha_k} c_\lambda) + \epsilon \sin\gamma (c_{\alpha_k} - c_\lambda)\sqrt{c_{\alpha_k}^2 + \sin^2\gamma - d_{\alpha_k}^2}}{c_{\alpha_k}^2 + \sin^2\gamma}. \tag{22}$$

Let A be as above and let A' be its quotient by relations $L_{\alpha\beta} = 0$ for some (α, β) pairs. Then $(A', K(\theta))$ is a solution of (1)–(5) with $f_\alpha(\theta)$ satisfying weaker conditions than those in theorem 1. In the extreme case—$L_{\alpha\beta} = 0$ for all (α, β), and no constraints on w_α besides (17)—we reproduce the trivial solution $[U_1, \chi^d] = 0$.

Note that (5) implies $y_\alpha(0) = \pm 1$. The $+1$ value is realized for all values of d_β except when $d_\beta = -c_\beta$, and then $y_\alpha(0) = -1$. For example, choosing $d_{\alpha_k} = -c_{\alpha_k}$ and $\epsilon = 1$ gives $d_\alpha = -c_\alpha$ for all $\alpha \in S$ (see (22)), and leads to a $K(0)$ which is not proportional to unity.

We next set up the mechanism to determine explicit matrix solutions.

Here we show that BH_n^d is generically isomorphic to a finite dimensional quotient of the affine Hecke algebra of type A_n, whose structure is known.

Definition 2. For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\beta)$ a d-tuple of indeterminates, define χ^d as the commutative algebra over $\mathbb{C}(\lambda)$ with generator X obeying: $\prod_{\alpha=1}^{d}(X - \lambda_\alpha) = 0$.

The algebras χ^d and χ^d are isomorphic via

$$X \mapsto \sum_{\alpha=1}^{d} \lambda_\alpha v_\alpha. \tag{23}$$

A specialization of χ^d is not isomorphic to χ^d unless the images of λ_α are distinct.

Definition 3 (generic affine Hecke quotient). Let $G_i = qU_i - 1$, then $E_n^d = E_n(q, \lambda)$ is the algebra generated by $\langle H_n, \chi^d \rangle / \rho$ where ρ is:

$$[X, G_i] = 0 \qquad \forall i > 1 \tag{24}$$

$$[G_1 X G_1, X] = 0. \tag{25}$$

Note that the representation theory of E_n^d is known, being determined by continuity with the wreath product $\mathbb{Z}_d \wr S_n$ [9], or by generalizing the B_n-type Hecke algebra [10].

Proposition 1. For w a collection of indeterminates let $Q_n^d(q, w)$ be a quotient of $\langle H_n, \chi^d \rangle$ obeying (18) and relations such that $H_2 \chi^d H_2 \chi^d \geqslant \chi^d H_2 \chi^d H_2$ as a vector space, and such that there exists a specialization

$$Q_2^d(1, w_0) \cong \langle X, G_1 : X^d = 1, [G_1 X G_1, X] = 0 \rangle. \tag{26}$$

Then $Q_n^d(q, w)$ is unique up to isomorphism.

The conditions for Q_n^d are satisfied by both BH_n^d and E_n^d. For example, $BH_2^d(1, a_{ij} = 1)$ gives the specialization (26) (cf (23)). We therefore obtain:

Corollary 1.1. The generic algebras BH_n^d and E_n^d are isomorphic.

An explicit isomorphism between $E_n^d(q, w)$ and the $BH_n^d(q, a)$ specialization defined by theorem 1 in the case where $k = d - 1$ is given by equation (23) and by

$$c_\alpha = -i \sin \gamma \frac{\lambda_1 + \lambda_\alpha}{\lambda_1 - \lambda_\alpha}. \tag{27}$$

We end with an example of a concrete matrix solution of (1)–(5). Let $R : H_n \mapsto \mathrm{End}((\mathbb{C}^N)^{\otimes n})$ be the fundamental vertex model representation of H_n [11]. Then $R(U_i)$ acts on $(\mathbb{C}^N)^{\otimes n}$ as $1 \otimes 1 \otimes \cdots \otimes \mathcal{U} \otimes \cdots \otimes 1$ where $\mathcal{U}$ is in the ith position and is given by:

$$\mathcal{U} = (q + q^{-1}) \sum_{i=1}^{N} E_{ii} \otimes E_{ii} + \sum_{i \neq j}^{N} E_{ij} \otimes E_{ji} + q \sum_{i<j}^{N} E_{ii} \otimes E_{jj} + q^{-1} \sum_{i>j}^{N} E_{ii} \otimes E_{jj} \tag{28}$$

where E_{ij} are elementary matrices. To satisfy $[U_i, X] = 0$ for $i > 1$ we take $R(\chi)$ to act non-trivially only on the first $\mathbb{C}^N$ of the tensor product. To keep our example simple, we look for a diagonal $R(\chi)$. We find that $v_\alpha = E_{\alpha\alpha} \otimes 1 \otimes 1 \cdots \otimes 1$, satisfy relations (19) provided that the parameters take the value $a_{\alpha\beta} = q$ for all $\alpha < \beta$. It is now possible to construct the full $K(\theta)$ corresponding to this representation. One finds that this $K(\theta)$ is non-degenerate only for $d = 2$.

We will conclude with a summary and discussion. We have defined an extension, BH_n^d, of the Hecke algebra H_n and shown that certain specializations of it give algebraic solutions to (1)–(5). We have also shown that BH_n^d and the affine Hecke algebra quotient E_n^d are generically isomorphic. This result provides a complete source of generic BH_n^d representations which can be used to construct matrix solutions to (1)–(5). Further physically motivated conditions on $K(\theta)$ (see [2,7]), can be incorporated as constraints on acceptable representations. Work in this area is in progress. We note that the same affine Hecke algebra also provides representations for integrable statistical mechanics models with periodic boundary conditions [8].

The next step in this work is to classify solutions associated with non-faithful representations (i.e. quotients) of BH_n^d. It would also be interesting to study extensions χ of H_n which are not of the form χ^d (the most general 6-vertex solution in [5] provides an example of this type), and to construct algebraic solutions to the reflection equation for other, non-Hecke, bulk solutions.

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